Performance Based Learning and Assessment Task

Maximizing Light Through a Window

I. ASSESSSMENT TASK OVERVIEW & PURPOSE:
This task is designed to develop skills in using the properties of quadratic functions in connection with concepts of area.

II. UNIT AUTHOR:
Mirela Geagla, Arlington Mill High School, Arlington Public Schools

III. COURSE:
Algebra 2 and Algebra Functions and Data Analysis

IV. CONTENT STRAND:
Algebra and Functions

V. OBJECTIVES:
The student will be able to:

- Develop a plan to determine the maximum amount of light
- Make connections between the problem and the quadratic function
- Find a rule for the quadratic function
- Find the maxima of quadratic function
- Find the dimensions of the window

VI. REFERENCE/RESOURCE MATERIALS:
Graphing calculators

VII. PRIMARY ASSESSMENT STRATEGIES:
The task includes an assessment component that performs two functions: (1) for the student it will be a checklist and provide a self-assessment and (2) for the teacher it will be used as a rubric. Include descriptions of any necessary accommodations.
The students/teacher will use a rubric for the mathematical content of the task. Students can use it to self-evaluate, and teachers to assign grades.
The students/teacher will use a rubric for the communication component of the task. Students will be able to self-evaluate their group work.
The students/teacher will use the Assessment List as a checklist. Students can use it to self-evaluate and teachers to assign a grade.

VIII. EVALUATION CRITERIA:
Mathematical content: rubric titled Problem Solving Rubric
Communication skills: rubric titled Group Task Rubric
Checklist: Assessment List
Benchmark of exemplary work
Problem-solving skills: checklist titled Math Processes Checklist

INSTRUCTIONAL TIME:
One 90 minutes class

Maximizing light through a window

Strand
Algebra, Functions

Mathematical Objective(s)
- Students will be able to relate the concepts of perimeter, circumference and area to develop a quadratic function in terms of the radius of the window.
- Students will apply knowledge of properties of quadratic functions to find the vertex of the function and its maxima.
- Students will use the values found for vertex and its corresponding y-value and apply them within the context of the problem in order to determine the dimensions of the window.

Related SOL
- AII.7 (investigate and analyze functions algebraically and graphically)
- AFDA.1 b) (The student will investigate and analyze quadratic functions, including local maxima and minima)

NCTM Standards
- Understand relations and functions and select, convert flexibly among, and use various representations for them.
- Apply and adapt a variety of appropriate strategies to solve problems.
- Communicate mathematical thinking coherently and clearly to peers, teachers, and others.

Materials/Resources
Classroom set of graphing calculator

Assumption of Prior Knowledge
- Students must have knowledge of quadratic functions, and to use formulas and calculators to determine the vertex and its maxima or minima.
- Students must be comfortable performing calculations with polynomials.
- Students will need to manipulate formulas for area, circumference, and perimeter.
- Students might begin to try to use numerical representations for the dimensions of the window (respectively for radius, width and length); consequently they might try to use those numbers to compute the area of the window.
- Students might feel inclined to repeat this process and compare the areas they found.
- Students might find it difficult to work with dimensions without knowing what they are.
- Students might find it difficult to recognize that the length of the rectangle is closely related to the radius of the circle.

Introduction: Setting Up the Mathematical Task

Description of the activity:

You are given the task of designing a window that gives the maximum amount of light. You have 10 feet of length to work with and find the dimensions of the window. The window is shaped like an arch, using a semicircle that shares its diameter with the length of a rectangle.

The teacher can introduce the activity by saying: "In this activity, you will investigate how to build a window so that it gives the maximum amount of light. You will be able to use a length of 10 feet for the enclosure of the window and apply that knowledge to answer the question. We know that the window is shaped like an arch, using a semicircle that shares its' diameter with the length of a rectangle."

- The teacher will prompt students to make a sketch of the window and label its' dimensions.
- The teacher then can ask students to think about what it means that the total given length is 10 feet.(prior knowledge assumed being concepts of circumference and area)
- The teacher will prompt students to explore different dimensions of the window; the teacher can ask: "What are the dimensions of a window with a radius of 1 foot?" "How does this knowledge affect the length and width of the rectangle?" "What do you need to know in order to find the amount of light that comes through the window?"
- If needed, more examples can be explored to allow for understanding of the relationship between the dimensions of the window and the total given length of 10 feet, and how that affects the area of the window.
- The teacher will prompt students to relate the dimensions of the rectangle to the radius of the circle, using appropriate variables; struggling students can use the aforementioned examples to develop this relationship.
- The teacher can prompt students to express the area of the window.
- The teacher will ask students to look for meaning in the expression of the area and relate that to their knowledge of functions.
- The students can work in pairs or groups of three.
- The students will share after the first exploration; comments such as "the radius has to be smaller than 2 feet", "if we know the radius, then we know the length of the rectangle, and the width", " the amount of light will be found by calculating area" should be heard during whole-class sharing.
- The students will share their findings upon completion of the task; a discussion will take place let by the teacher. The teacher can select a few groups to describe their strategies.

Student Exploration

Small Group Work – part I (estimated time:20 minutes)

- The students will work in groups to discuss how to approach the problem; they will start with making a sketch.
- If needed and prompted by the teacher, the students will explore possible dimensions of the window and how they affect the area; while doing this, the students should acknowledge the relationship between the dimensions of the rectangle and the radius of the circle.
- Each group will be able to find different areas and from that get an idea on how to determine the maximum amount of light (maximum area).

Whole Class Sharing/Discussion (10 minutes)

- Upon finding different areas, students will share their findings with the whole class; ideally there will be plenty of different examples that will give students an idea on how to further proceed.

Small Group Work – part II (estimated time: 45 minutes)

- The students will determine the variables (x for the radius of the circle, and y for the area), and express the area in terms of x (radius); the area is a quadratic expression in terms of the radius of the circle.
- The students will graph the quadratic function and find its maxima; the value of the maximum is reached for a specific value of x that students will find.
- The students will finish the task by concluding what all dimensions of the window should be in order to maximize the amount of light.

Whole Class Sharing/Discussion (15 minutes)

- The groups will share their findings with the whole class; special emphasis will be placed on what the variables are and what they represent.
- The teacher will encourage students to express their answers using mathematical vocabulary like: quadratic function, vertex, and maximum.
- The teacher can choose one or two groups to explain how they found their answer.

Student/Teacher Actions:

- Students should sketch the window using the information given at the beginning of the task. Some students might struggle, and if so, the teacher can help them understand what type of shapes the window is made of.
- The students should explore finding the total area of the window by using their own examples or those given by the teacher.
- The students might not know how to approach the task, and not know what to do; if so, the teacher can ask them to sketch a window that has a given radius of the circle of 1 foot. The students can then find the other dimensions of the window, and consequently the total area of the window.
- While exploring different sets of radii, the teacher should walk around the classroom, ensuring that students make a connection between the radius of the circle and the dimensions of the rectangle, especially the length (which is shared with the diameter of the semicircle).
- Possible questions to pose by the teacher: “What does it mean that the diameter is sharing its’ diameter with the length of the rectangle?”, “How does this affect the dimensions of the rectangle?”, “What are we looking to find since the requirement is the maximum amount of light?”, “What do you need to use in order to find the area of the window?”, “How do you use the equation of area to answer the question?”.
- The students should identify the two variables (x for the radius and y for the area), find the equation of the area, and consequently use that equation to determine the functions’ maxima.
- The students then should use the value of the maxima to find the dimensions of the window; the conclusion should be that the radius must be equal to the width/height of the window in order to maximize the amount of light.
- The teacher should monitor progress closely since the calculations for the area are a little cumbersome.
- The teacher will hold a whole-class discussion upon completion of task. The students will be able to address any issues they stumbled upon and how they were overcome.
- Time will be given at the end to explain how the dimensions were found and to discuss the struggles the students encountered and overcome.

Monitoring Student Responses

- The students will use the sketch and knowledge of perimeter and circumference to make a connection between the dimensions of the window and the total given length of 10 feet; this can be done by writing an equation.
- The students should clearly display how their use of formulas for areas of circle and rectangle led them to the area of the window; the equation for the area should only have one variable.
- The teacher can assist students who need help with their formulas by writing them on a post-it note.

- The students should be able to conclude that they must find an expression for the width of the rectangle in terms of the radius and the total enclosure of the window; without this expression, the equation for the area would contain two variables.
- The students should then express the area in terms of the width of the rectangle; this expression will become the quadratic function students should use to find its' maxima, and consequently answer the question.
- There will be two opportunities for sharing: one after students explored a few numerical examples, and another one upon completion of the task; this will give students opportunities to discuss the different struggles they encountered and what they did to overcome them. This will also provide an opportunity to reinforce the concept of maxima of a function and how it was used in this real world context.

Assessment List and Benchmark

Performance Task Assessment List

	0	2.5	5	Score
Student makes sketch of window	The student did not make a sketch of the window	The student made a sketch that is incorrect	The student made a correct sketch of the window	___/5
Student explores numerical examples	The student does not make any attempt to explore numerical examples	The student completes only one numerical example	The student completes at least two numerical examples	___/5
Student's use of formulas for circumference and perimeter of the window	The student has made more than 3 errors	The student has made 1 or 2 errors	The student uses formulas correctly without making any errors	___/5
Student's use of formulas for area of the window	The student has made more than 3 errors	The student has made 1 or 2 errors	The student uses formulas correctly without making any errors	___/5
Student writes equation for the area of the window in terms the radius of the circle	The student wrote an equation that is incorrect and the premises that led to this equation are incorrect as well	The student wrote an equation that has incorrect coefficients, but the steps that led to equation are correct	The student wrote the equation correctly	___/5
Student's use of the equation of the quadratic function	The student did not use the quadratic function to find its maximum value	The student found the maximum value by looking at the graph and making an approximation	The student found the maximum value by using a formula or the graphing calculator and its CALC features	___/5
Student's calculations of the dimensions of the window	The student did not find the dimensions of the window and/or the calculations are incorrect	The student found one out of the three dimensions and/or has one or two errors	The student found all the dimensions of the window and they are all correct	___/5

Math Processes Checklist

Student name:

	Consistently, Usually, Rarely/Never	**Dates Observed**	**Notes**
Student determines the problem.			
Student asks appropriate questions.			
Student chooses appropriate strategies.			
Student checks answers.			
Student justifies answers.			
Student sees relationships among concepts.			
Student applies strategies effectively.			
Student uses more than one strategy to solve a problem.			
Student uses mental math.			
Student plans and revises.			
Student shows work.			
Student generalizes processes to other situations.			
Student demonstrates task commitment.			

Group Task Rubric

Student name:

4	3	2	1
Understanding of Task			
I/we demonstrated an in-depth understanding of the content, processes, and demands of the task.	I/we demonstrated substantial understanding of the content and task, even though some supporting ideas or details may be overlooked or misunderstood.	I/we demonstrated gaps in our understanding of the content and task.	I/we demonstrated minimal understanding of the content.
Completion of Task			
I/we fully achieved the purpose of the task, including thoughtful, insightful interpretations and conjectures.	I/we accomplished the task.	I/we completed most of the assignment.	I/we attempted to accomplish the task, but with little or no success.
Communication of Findings			
I/we communicated our ideas and findings effectively, raised interesting and provocative questions, and went beyond what was expected.	I/we communicated our findings effectively.	I/we communicated our ideas and findings.	I/we did not finish the investigation and/or were not able to communicate our ideas very well.
Group Process			
We used all of our time productively. Everyone was involved and contributed to the group process and product.	We worked well together most of the time. We usually listened to each other and used each other's ideas.	We worked together som Not everyone contributed to the task.	We really did not pull together or work very productively as a group. Not everyone contributed to the group effort.

4	3	2	1
Problem Solving			
Problems did not deter us. We were proactive and worked together to solve problems.	We worked together to overcome problems we encountered.	We might have worked n productively as a group.	Some people did more wor than others. OR Nobody worked very well the group.

Maximizing Light through a Window

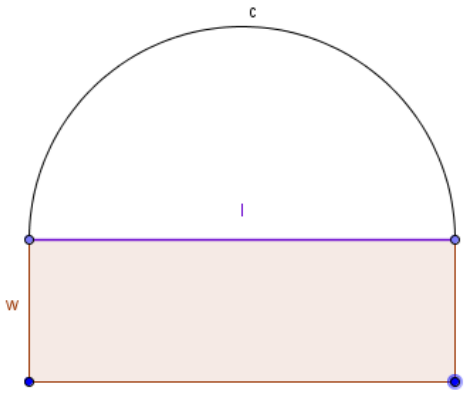
You are given the task of finding the dimensions of a window that gives the maximum amount of light. The window is shaped like an arch with a semicircle that shares its diameter with the length of a rectangle. You have a total of 10 feet of material to surround this window in its entirety. Find all dimensions of the window and explain your reasoning.

Maximizing Light through a Window

Sample of exemplary work

Small group work part I:

1. if r = 1 foot



$10 = w + w + 2 + \pi \bullet 1$

$10 = 2w + \pi + 2$

$10 - \pi - 2 = 2w$

$w = \dfrac{8 - \pi}{2}$

And l=2

In this case the area of the window is:

$$A = \frac{\pi r^2}{2} + l \bullet w = \frac{\pi \bullet 1^2}{2} + 2 \bullet \frac{8 - \pi}{2} = \frac{\pi}{2} + 8 - \pi = 8 - \frac{\pi}{2}\, ft^2 = 6.43\, ft^2$$

2. if r = 2 feet

$10 = w + w + 4 + \pi \bullet 2$

$10 = 2w + 4 + 2\pi$

$10 - 4 - 2\pi = 2w$

$w = 3 - \pi \approx -0.14\, ft$

Since the width of the window cannot be negative it concludes that the radius has to be smaller than 2 feet.

Small group work part II:

$10 = w + w + 2r + \pi r$

$10 = 2w + 2r + \pi r$

$10 - 2r - \pi r = 2w$

$w = \dfrac{10 - 2r - \pi r}{2}$

$$A = \frac{\pi r^2}{2} + (2r)\left(\frac{10 - 2r - \pi r}{2}\right) = \frac{\pi r^2}{2} + \frac{20r - 4r^2 - 2\pi r^2}{2} = \frac{\pi r^2 + 20r - 4r^2 - 2\pi r^2}{2} =$$

$$\frac{-\pi r^2 - 4r^2 + 20r}{2} = -\frac{\pi + 4}{2} r^2 + 10r$$

$$A(r) = -\frac{\pi + 4}{2} r^2 + 10r = -3.57 r^2 + 10r$$

When we use the calculator to find the maximum of the quadratic function above the maximum is:

$y \approx 7\ ft^2$ and $x \approx 1.4\ ft$.

The dimensions of the window are as follows:

$r \approx 1.4\ ft$

$l \approx 2.8\ ft$

$$w = \frac{10 - 2r - \pi r}{2} \approx \frac{10 - 2(1.4) - 3.14(1.4)}{2} = 1.4\ ft$$

So the maximum amount of light is given when both the height and radius of the circle are equal to 1.4 feet.

Different method of calculating the vertex for the function:

$$A(r) = -\frac{\pi + 4}{2} r^2 + 10r$$

Using the formula for the vertex of a quadratic function: $x = -\frac{b}{2a}$.

$$a = -\frac{\pi + 4}{2}, b = 10$$

$$x = -\frac{10}{2 \bullet -\frac{\pi + 4}{2}} = \frac{10}{\pi + 4}$$

So the radius of the circle is $\frac{10}{\pi + 4} \approx 1.4$, when the area is at a maximum value.

The length of the window is:

$$l = 2 \bullet r = 2 \bullet \frac{10}{\pi + 4} = \frac{20}{\pi + 4} \approx 2.8$$

The height of the window is:

$$w = \frac{10 - 2r - \pi r}{2} = \frac{10 - 2 \bullet \frac{10}{\pi + 4} - \pi \bullet \frac{10}{\pi + 4}}{2} = \frac{5}{1} - \frac{10}{\pi + 4} - \frac{5\pi}{\pi + 4} = \frac{5\pi + 20 - 10 - 5\pi}{\pi + 4} = \frac{10}{\pi + 4}$$

In conclusion, the height of the window is the radius of the circle when the maximum amount of light comes through.